Lecture 9

Proofs by Contraposition (contd.), Proof by Contradiction

Examples: Proof by Contraposition

Theorem: Suppose $x \in Z$. If $x^2 - 6x + 5$ is even, then x is odd.

Proof: We will prove the contrapositive of the theorem. That is,

Suppose $x \in Z$. If x is even, then $x^2 - 6x + 5$ is odd.

By the definition of an even integer,

If x is an even integer, then $x = 2k$, where k is some integer.

So, $x^2 - 6x + 5 = (2k)^2 - 6.(2k) + 5$

$$= 4k^2 - 12k + 5$$

$$= 4k^2 - 12k + 4 + 1$$

$$= 2(2k^2 - 6k + 2) + 1$$

$= 2k' + 1$, where k' is the integer $2k^2 - 6k + 2$.

Thus, $x^2 - 6x + 5$ is odd. ■

Examples: Proof by Contraposition

Theorem: Suppose $n \in \mathbb{Z}^+$. If $\underbrace{n \,\%\, 4 \text{ is } 2 \text{ or } 3}_{p}$, then $\underbrace{n \text{ is not a perfect square}}_{q}$.

$$\neg p = n \,\%\, 4 \text{ is neither } 2 \text{ nor } 3.$$
$$= n \,\%\, 4 \text{ is } 0 \text{ or } 1.$$

$$\neg q = n \text{ is a perfect square.}$$

Equivalent statements.

Theorem: Suppose $n \in \mathbb{Z}^+$. If n is a perfect square, then $n \,\%\, 4$ is 0 or 1.

Examples: Proof by Contraposition

Theorem: Suppose $n \in \mathbb{Z}^+$. If $n \,\%\, 4$ is 2 or 3, then n is not a perfect square.

Proof: We will prove the contrapositive of the theorem. That is,

Suppose $n \in \mathbb{Z}^+$. If n is a perfect square, then $n \,\%\, 4$ is 0 or 1.

Since n is a perfect square, $n = k^2$, where k is some integer.

There are four cases to consider, based on the value of $k \,\%\, 4$.

Case 1: When $k \,\%\, 4 = 0$

If $k \,\%\, 4 = 0$, then $k = 4q$, for some integer q.

Therefore, $n = k^2 = (4q)^2 = 4(4q^2)$. Hence, $n \,\%\, 4 = 0$.

Case 2: When $k \,\%\, 4 = 1$

If $k \,\%\, 4 = 1$, then $k = 4q + 1$, for some integer q.

Therefore, $n = k^2 = (4q + 1)^2 = 4(4q^2 + 2q) + 1$. Hence, $n \,\%\, 4 = 1$. …

Examples: Proof by Contraposition

Case 3: When $k \,\%\, 4 = 2$

If $k \,\%\, 4 = 2$, then $k = 4q + 2$, for some integer q.

Therefore, $n = k^2 = (4q + 2)^2 = 16q^2 + 16q + 4 = 4.(4q^2 + 4q + 1)$.

Hence, $n \,\%\, 4 = 0$.

Case 4: When $k \,\%\, 4 = 3$

If $k \,\%\, 4 = 3$, then $k = 4q + 3$, for some integer q.

Therefore, $n = k^2 = (4q + 3)^2 = 16q^2 + 24q + 9 = 4(4q^2 + 6q + 2) + 1$

Hence, $n \,\%\, 4 = 1$. ∎

Proof of Biconditional Statements

Because,

$$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$$

Proving both "if p, then q" and "if q, then p", proves "p if and only if q".

Example: Prove that "For an integer n, n is odd if and only if n^2 is odd."

We can prove the above statement by proving the below statements:

1. For an integer n, if n is odd, then n^2 is odd.
2. For an integer n, if n^2 is odd, then n is odd.

Proof by Contradiction

Outline of **Proof by Contradiction**.

1. The proposition to be proved is p.
2. We show that $\neg p$ implies falsehood. That is, proposition $\neg p \rightarrow q$ is true, where q is false.
   - $\neg p \rightarrow q$ is proven true by assuming $\neg p$ is true and then using it to prove q is also true.
   - Typically, q is of the form $r \wedge \neg r$.
3. Since $\neg p \rightarrow q$, where q is false, can be true only when $\neg p$ is false, we can conclude that p is true.

Note: Typically, q or r are not known in the beginning of the proof. We assume $\neg p$ and start deducing statements until we deduce some proposition r and $\neg r$.

Examples: Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof: Let $p = \sqrt{2}$ is irrational. Then, $\neg p = \sqrt{2}$ is rational.

Suppose $\neg p$ is true, i.e., $\sqrt{2}$ is rational. Then,

$$\sqrt{2} = \frac{a}{b} \qquad (1)$$

where $b \neq 0$, and a and b have no common factors.

Square on both sides of (1),

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

By the definition of an even integer, it follows that a^2 is even.

…

Examples: Proof by Contradiction

If a^2 is an even integer, then a is also even. Thus $a = 2k$, for some integer k.

Replace a with $2k$ in $2b^2 = a^2$. We get,

$$2b^2 = 4k^2$$
$$b^2 = 2k^2$$

By the definition of an even integer, it follows that b^2 is even. Therefore, b is also even.

We have shown that both a and b are even.

Therefore, a and b have a common factor, i.e., 2. But, we also deduced earlier that a and b have no common factor.

r $\neg r$

Because, $\neg p = \sqrt{2}$ is rational implies both "a and b have no common factor" and "a and b have a common factor", $\neg p$ must be false. Thus, p, i.e., $\sqrt{2}$ is irrational, must be true. ∎